

Network Coding for $3s/nt$ Sum-Networks

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Abstract—A sum-network is a directed acyclic network where each source independently generates one symbol from a given field \mathbb{F} and each terminal wants to receive the sum (over \mathbb{F}) of the source symbols. For sum-networks with two sources or two terminals, the solvability is characterized by the connection condition of each source-terminal pair [3]. A necessary and sufficient condition for the solvability of the 3-source 3-terminal ($3s/3t$) sum-networks was given by Shenvi and Dey [6]. However, the general case of arbitrary sources/sinks is still open. In this paper, we investigate the sum-network with three sources and n sinks using a region decomposition method. A sufficient and necessary condition is established for a class of $3s/nt$ sum-networks. As a direct application of this result, a necessary and sufficient condition of solvability is obtained for the special case of $3s/3t$ sum-networks.

I. INTRODUCTION

Network coding allows intermediate nodes of a communication network to combine the incoming information before forwarding it, and was shown to have significant throughput advantages as opposed to the conventional store-and-forward scheme [1], [2].

Most of the existent works of network coding focus on how the terminal nodes recover the whole or part of the original messages. Recently, network coding for communicating the sum of source messages to the terminal nodes was investigated [3]–[8]. Such a network is called as a sum-network. The problem of communicating sums over networks is in fact a subclass of the problem of distributed function computation, which has been considered in different contexts [9]–[12].

It was shown in [3] that for directed acyclic graphs with unit capacity edges and independent, unit-entropy sources, if there are two sources or two terminals in the network, then the network is solvable if and only if every source is connected to every terminal. For the 3-source 3-terminal ($3s/3t$) sum-networks, a necessary and sufficient condition for the solvability over any field is given in [6]. However, for networks with arbitrary number of sources and terminals, no necessary and sufficient condition is known.

In this paper, we consider the sum-networks with three sources using the technique of region decomposition [14], [15]. We give a necessary and sufficient condition for the solvability of a subclass of $3s/nt$ sum-networks. As a result, we give a simple characterization of solvability for the special case of $3s/3t$ sum-networks.

This paper is organized as follows. In Section II, we introduce the network model and the notations. The methodology is

proposed in section III. The main result is presented in Section IV. The paper is concluded in Section V.

II. MODELS AND NOTATIONS

We consider a directed, acyclic, finite graph $G = (V, E)$ with a set of k sources $\{s_1, \dots, s_k\}$ and a set of n terminals (sinks) $\{t_1, \dots, t_n\}$. Each source s_i generates a message $X_i \in \mathbb{F}$ and each terminal t_j wants to get the sum $\sum_{i=1}^k X_i$, where \mathbb{F} is a finite field. We assume that each link is error-free, delay-free and can carry one symbol from the field in each use. We call such network as a ks/nt sum-network.

For a link $e = (u, v) \in E$, u is called the *tail* of e and v is called the *head* of e , and are denoted by $u = \text{tail}(e)$ and $v = \text{head}(e)$, respectively. We call e an incoming link of v (an outgoing link of u). For two links $e, e' \in E$, we call e' an *incoming link* of e (e an *outgoing link* of e') if $\text{tail}(e) = \text{head}(e')$. For any $e \in E$, denoted by $\text{In}(e)$ the set of incoming links of e .

To aid analysis, we assume that each source s_i has an imaginary incoming link, called the X_i *source link* (or a *source link* for short), and each terminal t_j has an imaginary outgoing link, called a *terminal link*. Note that the source links have no tail and the terminal links have no head. As a result, the source links have no incoming link. For the sake of convenience, if $e \in E$ is not a source link, we call e a *non-source link*.

Let \mathbb{F}^k be the k -dimensional vector space over the finite field \mathbb{F} . For any subset $A \subseteq \mathbb{F}^k$, let $\langle A \rangle$ denote the subspace of \mathbb{F}^k spanned by A . For $i \in \{1, \dots, k\}$, we let α_i denote the vector of \mathbb{F}^k with the i th component being one and all other components being zero. Meanwhile, we let $\bar{\alpha} = \sum_{i=1}^k \alpha_i = (1, 1, \dots, 1)$, i.e., the vector with all components being one.

For any linear network coding scheme, the message along any link e is a linear combination $M_e = \sum_{i=1}^k c_i X_i$ of the source messages and we use the corresponding coding vector $d_e = (c_1, \dots, c_k)$ to represent the message, where $c_i \in \mathbb{F}$. To ensure the computability of network coding, the outgoing message, as a k -dimensional vector, must be in the span of all incoming messages. Moreover, to ensure that all terminals receive the sum $\sum_{i=1}^k X_i$, if e is a terminal link of the sum-network, then $d_e = \sum_{i=1}^k \alpha_i = \bar{\alpha}$. Thus, we can define a linear network code of a ks/nt sum-network as follows:

Definition 2.1 (Linear Network Code): Let $G = (V, E)$ be a ks/nt sum-network. A *linear code* (LC) of G over the field \mathbb{F} is a collection of vectors $C = \{d_e \in \mathbb{F}^k; e \in E\}$ such that

- (1) $d_e = \alpha_i$ if e is the X_i source link ($i = 1, \dots, k$);

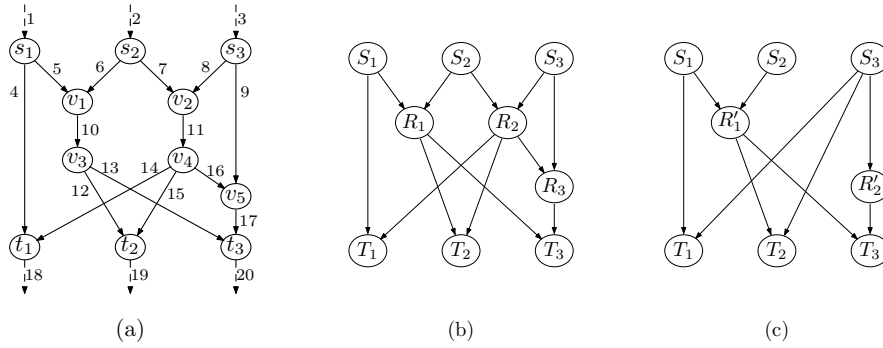


Fig 1. Examples of region graph: (a) is a 3s/3t sum-network G_1 , where all links are sequentially indexed as $1, 2, \dots, 20$. Here, the imaginary links $1, 2, 3$ are the X_1, X_2, X_3 source link, and $18, 19, 20$ are the terminal links at terminal t_1, t_2, t_3 respectively. (b) is the region graph $\text{RG}(D)$, where $S_1 = \{1, 4, 5\}, S_2 = \{2, 6, 7\}, S_3 = \{3, 8, 9\}, R_1 = \{10, 12, 13\}, R_2 = \{11, 14, 15, 16\}, R_3 = \{17\}, T_1 = \{18\}, T_2 = \{19\}, T_3 = \{20\}$ and $D = \{S_1, S_2, S_3, R_1, R_2, R_3, T_1, T_2, T_3\}$. (c) is the region graph $\text{RG}(D')$, where $S_1 = \{1, 4, 5\}, S_2 = \{2, 6, 7\}, S_3 = \{3, 8, 9\}, R'_1 = \{10, 12, 13\}, R'_2 = \{11, 14, 15, 16, 17\}, T_1 = \{18\}, T_2 = \{19\}, T_3 = \{20\}$ and $D' = \{S_1, S_2, S_3, R'_1, R'_2, T_1, T_2, T_3\}$.

(2) $d_e \in \langle d_{e'}; e' \in \text{In}(e) \rangle$ if e is a non-source link.

The code $C = \{d_e \in \mathbb{F}^k; e \in E\}$ is said to be a *linear solution* of G if $d_e = \bar{\alpha}$ for all terminal link e .

The vector d_e is called the *global encoding vector* of link e . The network G is said to be *solvable* if it has a linear solution over some finite field \mathbb{F} .

III. REGION DECOMPOSITION AND NETWORK CODING

In this section, we present the region decomposition approach, which will take a key role in our discussion. The basic idea of region decomposition is proposed in [14], [15].

A. Region Decomposition and Region Graph

Definition 3.1 (Region and Region Decomposition): Let R be a non-empty subset of E . R is called a region of G if there is an $e_l \in R$ such that for any $e \in R$ and $e \neq e_l$, R contains an incoming link of e . If E is partitioned into mutually disjoint regions, say R_1, R_2, \dots, R_N , then we call $D = \{R_1, R_2, \dots, R_N\}$ a region decomposition of G .

The edge e_l in Definition 3.1 is called the leader of R and is denoted as $e_l = \text{lead}(R)$. A region R is called the X_i *source region* (or a *source region* for short) if $\text{lead}(R)$ is the X_i source link; R is called a *terminal region* if R contains a terminal link. If R is neither a source region nor a terminal region, we call R a coding region. If R is not a source region, we call R a *non-source region*.

Since the source links have no incoming link, then each source region contains exactly one source link, i.e., its leader. But a terminal region may contains more than one terminal links. So there are exactly k source region and at most n terminal regions for any ks/nt sum-network. We will always denote the k source regions as S_1, \dots, S_k and the n terminal regions as T_1, \dots, T_n .

Definition 3.2 (Region Graph): Let D be a region decomposition of G . The region graph of G about D is a directed, simple graph with vertex set D and edge set \mathcal{E}_D , where \mathcal{E}_D is the set of all ordered pairs (R', R) such that R' contains an incoming link of $\text{lead}(R)$.

Consider the example network G_1 in Fig. 1 (a). Examples of two region graphs are shown in Fig. 1 (b) and (c). In general, G may have many region decompositions.

We use $\text{RG}(D)$ to denote the region graph of G about D , i.e., $\text{RG}(D) = (D, \mathcal{E}_D)$. If (R', R) is an edge of $\text{RG}(D)$, we call R' a parent of R . For $R \in D$, we use $\text{In}(R)$ to denote the set of parents of R in $\text{RG}(D)$. Since the source links have no incoming link, then the source regions have no parent. Moreover, since G is acyclic, then clearly, $\text{RG}(D)$ is acyclic.

For $R, R' \in D$, a path in $\text{RG}(D)$ from R' to R is a sequence of regions $\{R_0 = R', R_1, \dots, R_p = R\}$ such that R_{i-1} is a parent of $R_i, i = 1, \dots, p$. If there is a path from R' to R , we say R' is connected to R and denote $R' \rightarrow R$. Else, we say R' is not connected to R and denote $R' \nrightarrow R$. In particular, we regard $R \rightarrow R$ for all $R \in D$.

B. Network Coding on Region Graph

Definition 3.3 (Codes on Region Graph): A *linear code* (LC) of the region graph $\text{RG}(D)$ over the field \mathbb{F} is a collection of vectors $\tilde{C} = \{d_R \in \mathbb{F}^k; R \in D\}$ such that

- (1) $d_{S_i} = \alpha_i$, where S_i is the X_i source region for each $i \in \{1, \dots, k\}$;
- (2) $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle$ if R is a non-source region.

The code $\tilde{C} = \{d_R \in \mathbb{F}^k; R \in D\}$ is said to be a *linear solution* of $\text{RG}(D)$ if $d_{T_j} = \bar{\alpha}$ for each terminal region T_j .

The vector d_R is called the *global encoding vector* of R . The region graph $\text{RG}(D)$ is said to be *feasible* if it has a linear solution over some finite field \mathbb{F} .

By Definition 3.3, for any linear solution of $\text{RG}(D)$, it is always be that $d_{S_i} = \alpha_i$ and $d_{T_j} = \bar{\alpha}$. So in order to obtain a solution, we only need to specify the global encoding vector for each coding region.

Let D be a region decomposition of G . Clearly, any linear solution of $\text{RG}(D)$ can be extended to a linear solution of G by letting $d_e = d_R$ for each $R \in D$ and each $e \in R$. So if $\text{RG}(D)$ is feasible, then G is solvable. But conversely, if G is solvable, it is not necessary that $\text{RG}(D)$ is feasible.

For the region graph $\text{RG}(D)$ in Fig. 1 (b), let $d_{R_1} = \alpha_1$ and $d_{R_2} = d_{R_3} = \alpha_2 + \alpha_3$. Then $\tilde{C} = \{d_R; R \in D\}$ is a linear solution of $\text{RG}(D)$ and we can obtain a linear solution of G_1 by letting $d_e = d_R$ for each $R \in D$ and each $e \in R$. However, the region graph $\text{RG}(D')$ in Fig. 1 (c) is not feasible because

for any linear code, by conditions (1), (2) of Definition 3.3, $d_{T_1} \in \langle \alpha_1, \alpha_3 \rangle$. So it is impossible that $d_{T_1} = \bar{\alpha}$.

In the following, we shall define a special region decomposition D^{**} of G , called the basic region decomposition of G , which is unique and has the property that G is solvable if and only if the region graph $\text{RG}(D^{**})$ is feasible.

Definition 3.4 (Basic Region Decomposition[14]): Let D^{**} be a region decomposition of G . D^{**} is called a basic region decomposition of G if the following conditions hold:

- (1) For any $R \in D^{**}$ and any $e \in R \setminus \{\text{lead}(R)\}$, $\text{In}(e) \subseteq R$;
- (2) Each non-source region R in D^{**} has at least two parents in $\text{RG}(D^{**})$.

Accordingly, the region graph $\text{RG}(D^{**})$ is called a basic region graph of G .

For example, one can check that for the network G_1 in Fig. 1 (a), the region graph $\text{RG}(D)$ in Fig. 1 (b) is the basic region graph of G_1 .

The basic region decomposition D^{**} can be decided within time $O(|E|)$ (See Algorithm 5 in [14]. Note that this Algorithm can be generalized to networks with any k sources directly.). The following two theorems were also derived in [14] (See Theorem 4.4 and 4.5 of [14] respectively.) and we omit their proof.

Theorem 3.5: G has a unique basic region decomposition, hence has a unique basic region graph.

Theorem 3.6: G is solvable if and only if $\text{RG}(D^{**})$ is feasible, where D^{**} is the basic region decomposition of G .

C. Super Region

In this subsection, we always assume that D is a region decomposition of G such that each non-source region has at least two parents in $\text{RG}(D)$.

Definition 3.7 (Super Region [15]): Let D be a region decomposition of G and $\emptyset \neq \Theta \subseteq D$. The super region generated by Θ , denoted by $\text{reg}(\Theta)$, is defined recursively as follows:

- (1) If $R \in \Theta$, then $R \in \text{reg}(\Theta)$;
- (2) If $R \in D$ and $\text{In}(R) \subseteq \text{reg}(\Theta)$, then $R \in \text{reg}(\Theta)$.

We define $\text{reg}^\circ(\Theta) = \text{reg}(\Theta) \setminus \Theta$. Moreover, if $\Theta = \{R_1, \dots, R_k\}$, then we denote $\text{reg}(\Theta) = \text{reg}(R_1, \dots, R_k)$.

Since $\text{RG}(D)$ is acyclic, then $\text{reg}(\Theta)$ is well defined.

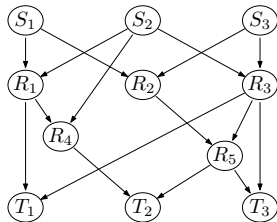


Fig. 2. An example of region graph.

Consider the region graph in Fig. 2. We have $\text{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$ and $\text{reg}(S_2, S_3) = \{S_2, S_3, R_2, R_3\}$.

Remark 3.8: From Definition 3.3 and 3.7, it is easy to see that if $\tilde{C} = \{d_R \in \mathbb{F}^k; R \in D\}$ is a linear code of $\text{RG}(D)$ and $\emptyset \neq \Theta \subseteq D$, then $d_R \in \langle d_{R'}; R' \in \Theta \rangle$ for all $R \in \text{reg}(\Theta)$.

Lemma 3.9: Suppose Θ_1 and Θ_2 are two subsets of D . Then $\text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) = \text{reg}(\Theta)$, where

$$\Theta = (\text{reg}(\Theta_1) \cap \Theta_2) \cup (\text{reg}(\Theta_2) \cap \Theta_1).$$

Proof: Clearly, $\Theta \subseteq \text{reg}(\Theta_1)$ and $\Theta \subseteq \text{reg}(\Theta_2)$. Then by Definition 3.7, we have $\text{reg}(\Theta) \subseteq \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2)$.

Now, suppose $\text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) \neq \text{reg}(\Theta)$. Then there is an R_0 such that

$$R_0 \in \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) \setminus \text{reg}(\Theta).$$

By assumption of Θ , we have $R_0 \notin \Theta_1 \cup \Theta_2$. (Otherwise, without loss of generality, assume $R_0 \in \Theta_1$. Then $R_0 \in (\text{reg}(\Theta_2) \cap \Theta_1) \subseteq \Theta \subseteq \text{reg}(\Theta)$, which contradict to the assumption of $R_0 \notin \text{reg}(\Theta)$.) So $R_0 \in \text{reg}^\circ(\Theta_1) \cap \text{reg}^\circ(\Theta_2)$. Then by Definition 3.7, we have

$$\text{In}(R_0) \subseteq \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2).$$

Since $R_0 \notin \text{reg}(\Theta)$, then by Definition 3.7, there exists an $R_1 \in \text{In}(R_0)$ such that $R_1 \notin \text{reg}(\Theta)$. Then

$$R_1 \in \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) \setminus \text{reg}(\Theta).$$

Similarly, R_1 has a parent R_2 such that

$$R_2 \in \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) \setminus \text{reg}(\Theta).$$

By repeating this process, we can find a series of infinite regions R_0, R_1, R_2, \dots such that R_i is a parent of R_{i-1} and

$$R_i \in \text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) \setminus \text{reg}(\Theta), \quad i = 1, 2, \dots$$

This contradicts to the fact that $\text{RG}(D)$ is a finite graph. So $\text{reg}(\Theta_1) \cap \text{reg}(\Theta_2) = \text{reg}(\Theta)$. ■

IV. A SUFFICIENT AND NECESSARY CONDITION FOR A SUBCLASS OF 3-SOURCE SUM-NETWORKS

Throughout this section, we always assume that G is a 3s/nt sum-network and D^{**} is the basic region decomposition of G . By Theorem 3.6, G is solvable if and only if $\text{RG}(D^{**})$ is feasible. So we only need to consider coding on $\text{RG}(D^{**})$.

Since G is a 3s/nt sum-network, then $\text{RG}(D^{**})$ has exactly three source regions and at most n terminal regions. Without loss of generality, we assume $\text{RG}(D^{**})$ has exactly n terminal regions. Let S_i ($i \in \{1, 2, 3\}$) denote the X_i source region and $T_j, j = 1, \dots, n$, denote the n terminal regions. For any $i \in \{1, 2, 3\}$, by Lemma 3.9, we have

$$\text{reg}(S_i, S_{j_1}) \cap \text{reg}(S_i, S_{j_2}) = \{S_i\} \quad (1)$$

where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$. Thus

$$\text{reg}^\circ(S_i, S_{j_1}) \cap \text{reg}^\circ(S_i, S_{j_2}) = \emptyset. \quad (2)$$

i.e., $\text{reg}^\circ(S_1, S_2)$, $\text{reg}^\circ(S_1, S_3)$ and $\text{reg}^\circ(S_2, S_3)$ are mutually disjoint. Denote $\{1, \dots, n\} = [n]$ for any positive integer n .

Definition 4.1: We define some subsets of D^{**} as follows:

- (1) $\Pi \triangleq \text{reg}(S_1, S_2) \cup \text{reg}(S_1, S_3) \cup \text{reg}(S_2, S_3)$;
- (2) For any $I \subseteq [n]$, Ω_I is the set of all $R \in D^{**} \setminus \Pi$ such that $R \rightarrow T_j, \forall j \in I$, and $R \nrightarrow T_{j'}, \forall j' \in [n] \setminus I$;

(3) Λ_I is the set of all $Q \in \Pi$ such that Q has a child $R \in \Omega_I$. We also denote $\Omega_I = \Omega_{i_1, \dots, i_p}$ and $\Lambda_I = \Lambda_{i_1, \dots, i_p}$ if the subset $I = \{i_1, \dots, i_p\}$.

From the above definition, the following remark is obvious.

Remark 4.2: If $I, I' \subseteq [n]$ and $I \neq I'$, then $\Omega_I \cap \Omega_{I'} = \emptyset$.

Since $\text{RG}(D^{**})$ is acyclic, regions in D^{**} can be sequentially indexed as $D^{**} = \{R_1, R_2, R_3, \dots, R_N\}$ such that $R_i = S_i, i = 1, 2, 3, R_{N-n+j} = T_j, j = 1, 2, \dots, n$, and $\ell < \ell'$ if R_ℓ is a parent of $R_{\ell'}$. For all $I \subseteq [n]$, we can determine Ω_I by the following simple labelling algorithm.

Algorithm 1: Labelling Algorithm ($\text{RG}(D^{**})$):

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j ← from 1 to n
Label  $R_{N-n+j}$  with j;
ℓ ← from N to 1;
if  $R_\ell$  has a child  $R_{\ell'}$  such that  $R_{\ell'}$  is labelled with j for
some  $j \in [n]$  then
    Label  $R_\ell$  with j;

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Note that for each $R \in D^{**}$ and $j \in [n]$, if R has a child R' such that $R' \rightarrow T_j$, then $R \rightarrow T_j$. So $R \rightarrow T_j$ if and only if R is labelled with j by Algorithm 1. Let $I_R = \{j \in [n]; R \text{ is labelled with } j\}$. Then for each $I \subseteq [n]$, $R \in \Omega_I$ if and only if $I_R = I$. Thus, by Algorithm 1, we can easily determine Ω_I for all $I \subseteq [n]$. Clearly, the run time of Algorithm 1 is $O(|D^{**}|)$.

Consider the region graph in Fig. 2. We have $\Omega_i = \{T_i\}$ for $i \in \{1, 2, 3\}$, $\Omega_{2,3} = \{R_5\}$ and $\Omega_{1,2} = \Omega_{1,3} = \Omega_{1,2,3} = \emptyset$. Thus, we have $\Lambda_1 = \{R_1, R_3\}$, $\Lambda_2 = \{R_4\}$, $\Lambda_3 = \{R_3\}$, $\Lambda_{2,3} = \{R_2, R_3\}$ and $\Lambda_{1,2} = \Lambda_{1,3} = \Lambda_{1,2,3} = \emptyset$.

If $\tilde{C} = \{d_R \in \mathbb{F}^k; R \in D\}$ is a linear solution of $\text{RG}(D^{**})$, $\{i, j\} \subseteq \{1, 2, 3\}$ and $R \in \text{reg}(S_i, S_j)$, then by Remark 3.8, $\bar{\alpha} \neq d_R \in \langle \alpha_i, \alpha_j \rangle$. So $T_\ell \notin \text{reg}(S_i, S_j)$ for all terminal region T_ℓ , which implies that $T_\ell \in D^{**} \setminus \Pi, \forall \ell \in [n]$. For this reason, in this section, we assume:

Assumption 1: $T_j \notin \Pi$ for all $j \in [n]$.

Definition 4.3: The region graph $\text{RG}(D^{**})$ is said to be terminal-separable if $\Omega_I = \emptyset$ for all $I \subseteq [n]$ such that $|I| > 1$.

For example, the region graph in Fig. 3 is terminal-separable. However, the region graph in Fig. 2 is not because $\Omega_{2,3} = \{R_5\} \neq \emptyset$.

We shall give a necessary and sufficient condition of feasibility for terminal-separable region graph, by which it is easy to check whether a terminal-separable region graph is feasible.

Remark 4.4: Terminal-separable region graphs is of interesting because, if a region graph $\text{RG}(D)$ is not terminal-separable, then we can view it as a region graph with fewer terminal regions. For example, for the region graph in Fig. 2, we can view T_1 and R_5 as two terminal regions and construct a linear solution of $\text{RG}(D)$. Then the sum of sources can be transmitted from R_5 to T_2 and T_3 . In fact, let $d_{R_1} = d_{R_2} = \alpha_1, d_{R_3} = \alpha_2 + \alpha_3$ and $d_{R_5} = \alpha_1 + \alpha_2 + \alpha_3$. Then $\{d_R; R \in D\}$ is a linear solution of $\text{RG}(D)$.

Lemma 4.5: If $\text{RG}(D^{**})$ is terminal-separable, then for all $j \in [n]$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, we have $T_j \in \Omega_j \subseteq \text{reg}^\circ(\Lambda_j)$ and $\Lambda_j \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$. In particular, we have $|\Lambda_j| \geq 2$.

Proof: Since $T_j \rightarrow T_j$ and $\text{RG}(D^{**})$ is terminal-separable, then $T_j \not\rightarrow T_{j'}, \forall j' \in [n] \setminus \{j\}$. So $T_j \in \Omega_j$.

We now prove $\Omega_j \subseteq \text{reg}(\Lambda_j)$ by contradiction. For this purpose, suppose there is an $R \in \Omega_j$ such that $R \notin \text{reg}(\Lambda_j)$. Then by Definition 3.7, R has a parent, say P_1 , such that $P_1 \notin \text{reg}(\Lambda_j)$. Clearly, $P_1 \notin \Pi$. (Otherwise, by the definition of Λ_j , $P_1 \in \Lambda_j \subseteq \text{reg}(\Lambda_j)$, which contradicts to the assumption that $P_1 \notin \text{reg}(\Lambda_j)$.) Since $\text{RG}(D^{**})$ is terminal-separable and $P_1 \rightarrow R \rightarrow T_j$, then $P_1 \not\rightarrow T_{j'}, \forall j' \neq j$. So $P_1 \in \Omega_j$. Similarly, P_1 has a parent P_2 such that $P_2 \notin \text{reg}(\Lambda_j)$ and $P_2 \in \Omega_j$. By repeating this process, we can obtain a series of infinite regions, P_1, P_2, \dots such that $P_i \notin \text{reg}(\Lambda_j)$ and $P_i \in \Omega_j$, which contradicts to the fact that $\text{RG}(D^{**})$ is a finite graph. So $\Omega_j \subseteq \text{reg}(\Lambda_j)$.

Note that $\Omega_j \subseteq D^{**} \setminus \Pi$ and $\Lambda_j \subseteq \Pi$. So $\Omega_j \cap \Lambda_j = \emptyset$. Thus, we have $T_j \in \Omega_j \subseteq \text{reg}^\circ(\Lambda_j)$.

Moreover, if $\Lambda_j \subseteq \text{reg}(S_{i_1}, S_{i_2})$, then by Definition 3.7, we have $T_j \in \Omega_j \subseteq \text{reg}^\circ(\Lambda_j) \subseteq \text{reg}(S_{i_1}, S_{i_2})$, which contradicts to Assumption 1. So $\Lambda_j \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$.

Finally, if $|\Lambda_j| = 1$, say $\Lambda_j = \{Q\}$, then by the definition of Π and Λ_j , we have $Q \in \text{reg}(S_{i_1}, S_{i_2})$ for some $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. Thus, $\Lambda_j = \{Q\} \subseteq \text{reg}(S_{i_1}, S_{i_2})$, which contradicts to the proved result that $\Lambda_j \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$. So $|\Lambda_j| \geq 2$. ■

Lemma 4.6: Suppose $\text{RG}(D^{**})$ is terminal-separable. Then $\text{RG}(D^{**})$ is feasible if and only if there is a collection of vectors $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfying the following three conditions:

- (1) $d_{S_i} = \alpha_i, i = 1, 2, 3$;
- (2) $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle, \forall R \in \Pi \setminus \{S_1, S_2, S_3\}$;
- (3) $\bar{\alpha} \in \langle d_R; R \in \Lambda_j \rangle, \forall j \in [n]$.

Proof: Suppose $\text{RG}(D^{**})$ is feasible and $\tilde{C} = \{d_R; R \in D^{**}\}$ is a linear solution of $\text{RG}(D^{**})$. By Lemma 4.5, $T_j \in \Omega_j \subseteq \text{reg}^\circ(\Lambda_j)$ and $\Lambda_j \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall j \in [n]$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. By Remark 3.8, $\bar{\alpha} = d_{T_j} \in \langle d_R; R \in \Lambda_j \rangle$. Let $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$. Then \tilde{C}_Π satisfies conditions (1)-(3).

Conversely, suppose there is a collection $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfying conditions (1)-(3). We can construct a linear solution of $\text{RG}(D^{**})$ as follows:

Since $\text{RG}(D^{**})$ is terminal-separable, for each $j \in [n]$ and $Q \in \Lambda_j$, by the Definition of Λ_j and Ω_j , we can find a path $\{R_1, \dots, R_\ell\} \subseteq \Omega_j$ such that $R_\ell = T_j$ and Q is a parent of R_1 . Let Γ_j be the union of all such paths. Then $\Gamma_j \subseteq \Omega_j$. Since $\bar{\alpha} \in \langle d_R; R \in \Lambda_j \rangle$, then we can construct a code $\tilde{C}_{\Gamma_j} = \{d_R; R \in \Gamma_j\}$ such that $d_{T_j} = \bar{\alpha}$ and $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle$ for all $R \in \Gamma_j$. By Remark 4.2, $\Omega_j, j = 1, \dots, n$, are mutually disjoint. So $\Gamma_j, j = 1, \dots, n$, are mutually disjoint and $\tilde{C} = \tilde{C}_\Pi \cup \tilde{C}_{\Gamma_1} \cup \dots \cup \tilde{C}_{\Gamma_n}$ is a linear solution of $\text{RG}(D^{**})$. Thus, $\text{RG}(D^{**})$ is feasible. ■

A. Partitioning of Π

To give a simple characterization of feasibility of $\text{RG}(D^{**})$, we need to make some discussion on partitioning Π .

Let $\mathcal{I} = \{\Delta_1, \dots, \Delta_K\}$ be a partition of Π . For the sake of convenience, we shall call each Δ_i an *equivalent class* of \mathcal{I} . If $R \in \Delta_i$, we denote $\Delta_i = [R]$. Thus, for each Δ_i , we can choose an $R_i \in \Delta_i$ and denote $\mathcal{I} = \{[R_1], \dots, [R_K]\}$.

Let $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ be an arbitrary partition of Π .¹ For each equivalent class $[R] \in \mathcal{I}$ and each subset $\{i, j\} \subseteq \{1, 2, 3\}$, we denote

$$[R]_{i,j} = [R] \cap \text{reg}(S_i, S_j). \quad (3)$$

For each $i \in \{1, 2, 3\}$, we denote

$$[S_i]_i = [S_i]_{i,j_1} \cup [S_i]_{i,j_2} \quad (4)$$

where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$. Then we can divide each equivalent class as follows:

Definition 4.7: For $i \in \{1, 2, 3\}$, $[S_i]$ is divided into two subclasses $[S_i]_i$ and $[S_i]_{j_1, j_2}$, where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$; For $i \in \{4, \dots, K\}$, $[R_i]$ is divided into three subclasses $[R_i]_{1,2}$, $[R_i]_{1,3}$ and $[R_i]_{2,3}$.

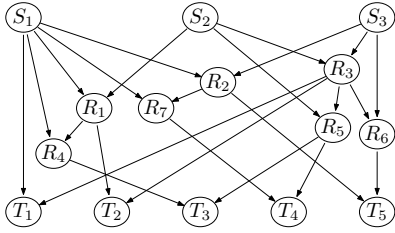


Fig. 3. An example of terminal-separable region graph: By Definition 3.7, we can check that $\text{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$, $\text{reg}(S_1, S_3) = \{S_1, S_3, R_2, R_7\}$ and $\text{reg}(S_2, S_3) = \{S_2, S_3, R_3, R_5, R_6\}$. By Definition 4.1, we have $\Omega_j = \{T_j\}, j = 1, \dots, 6$ and $\Omega_T = \emptyset, \forall I \subseteq \{1, \dots, 6\}$ such that $|I| \geq 2$. So this region graph is terminal-separable.

For any equivalent class $[R] \in \mathcal{I}$, we use $[[R]]$ to denote a subclass of $[R]$. Note that a subclass $[[R]]$ of $[R]$ is possibly an empty set. By Equations (1)–(4), each equivalent class is a disjoint union of its all subclasses. Thus, $\{[S_1]_1, [S_1]_{2,3}\} \cup \{[S_2]_2, [S_2]_{1,3}\} \cup \{[S_3]_3, [S_3]_{1,2}\} \cup (\cup_{i=4}^K \{[R_i]_{1,2}, [R_i]_{1,3}, [R_i]_{2,3}\})$ is still a partition of Π .

Example 4.8: Consider the region graph in Fig. 3. By Definition 3.7, $\text{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$, $\text{reg}(S_1, S_3) = \{S_1, S_3, R_2, R_7\}$ and $\text{reg}(S_2, S_3) = \{S_2, S_3, R_3, R_5, R_6\}$. Let $[S_1] = \{S_1, R_1, R_3, R_4, R_5, R_7\}$, $[S_2] = \{S_2\}$, $[S_3] = \{S_3\}$, $[R_2] = \{R_2, R_6\}$ and $\mathcal{I} = \{[S_1], [S_2], [S_3], [R_2]\}$. Then \mathcal{I} is a partition of Π and $[S_1]_1 = \{S_1, R_1, R_4, R_7\}$, $[S_1]_{2,3} = \{R_3, R_5\}$, $[S_2]_2 = \{S_2\}$, $[S_2]_{1,3} = \emptyset$, $[S_3]_3 = \{S_3\}$, $[S_3]_{1,2} = \emptyset$, $[R_2]_{1,2} = \emptyset$, $[R_2]_{1,3} = \{R_2\}$, $[R_2]_{2,3} = \{[R_6]\}$ are all subclasses of \mathcal{I} and they also form a partition of Π .

Definition 4.9: Let $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ be a partition of Π . Two equivalent classes $[R']$ and $[R'']$ are said to be *connected* if one of the following conditions hold:

- (1) There is a $j \in [n]$ such that $\Lambda_j \subseteq [[R']] \cup [[R'']]$, where $[[R']]$ (resp. $[[R'']]$) is a subclass of $[R']$ (resp. $[R'']$);
- (2) There is an $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ such that $\text{reg}([R']_{i_1, i_2}) \cap \text{reg}([R'']_{i_1, i_2}) \neq \emptyset$.

¹When we use the notation $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$, we always assume that $[S_1], [S_2], [S_3], \dots, [R_K]$ are mutually different.

Definition 4.10: Let $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ be a partition of Π . \mathcal{I} is said to be *compatible* if the following two conditions hold:

- (1) No pair of equivalent classes of \mathcal{I} are connected;
- (2) $\Lambda_j \not\subseteq [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup (\cup_{\ell=4}^K [R_\ell]_{i_1, i_2})$ for all $j \in [n]$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$.

Clearly, in Example 4.8, the partition \mathcal{I} of Π is compatible.

Suppose \mathcal{I} is a partition of Π and $\{[R'], [R'']\} \subseteq \mathcal{I}$. By combining $[R']$ and $[R'']$ into one equivalent class $[R'] \cup [R'']$, we obtain a partition $\mathcal{I}' = \mathcal{I} \cup \{[R'] \cup [R'']\} \setminus \{[R'], [R'']\}$ of Π . We call \mathcal{I}' a *contraction* of \mathcal{I} by combining $[R']$ and $[R'']$.

B. Main Result

Let $\mathcal{I}_0 = \{[R]; R \in \Pi\}$, where $[R] = \{R\}, \forall R \in \Pi$. Then \mathcal{I}_0 is a partition of Π . We call \mathcal{I}_0 the trivial partition of Π .

Definition 4.11: Let $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_L = \mathcal{I}_c$ be a sequence of partitions of Π such that \mathcal{I}_ℓ is a contraction of $\mathcal{I}_{\ell-1}$ by combining two connected equivalent classes in $\mathcal{I}_{\ell-1}$ and, for any $\{i, j\} \subseteq \{1, 2, 3\}$, $[S_i] \neq [S_j]$ in $\mathcal{I}_{\ell-1}$, where $\ell = 1, \dots, L$. \mathcal{I}_c is called a *character partition* of Π if one of the following conditions hold:

- (1) $[S_i] = [S_j]$ for some $\{i, j\} \subseteq \{1, 2, 3\}$;
- (2) No pair of equivalent classes in \mathcal{I}_c are connected.

Example 4.12: Consider the region graph in Fig. 4 (a). We have $\Pi = \{S_1, S_2, S_3, R_1, R_2, R_3, R_4\}$ and $\Lambda_2 = \{S_2, R_1\} \subseteq ([S_2]_2 \cup [R_1]_{1,3})$. By (1) of Definition 4.9, $[S_2]$ and $[R_1]$ are connected. So $\mathcal{I}_1 = \{\{S_1\}, \{S_2, R_1\}, \{S_3\}, \{R_2\}, \{R_3\}, \{R_4\}\}$ is obtained from \mathcal{I}_0 by combining $[S_2]$ and $[R_1]$, where \mathcal{I}_0 is the trivial partition of Π . Similarly, let $\mathcal{I}_2 = \{\{S_1\}, \{S_2, R_1, R_2\}, \{S_3\}, \{R_3\}, \{R_4\}\}$ and $\mathcal{I}_3 = \{\{S_1\}, \{S_2, R_1, R_2, R_3\}, \{S_3\}, \{R_4\}\}$. Then $\mathcal{I}_j, j = 2, 3$, is obtained from \mathcal{I}_{j-1} by combining two connected equivalent classes. Note that in \mathcal{I}_3 , $\text{reg}([S_2]_{2,3}) = \text{reg}(R_2, R_3) = \{R_2, R_3, R_4\}$ and $\text{reg}([R_4]_{2,3}) = \text{reg}(R_4) = \{R_4\}$. So by (2) of Definition 4.9, $[S_2]$ and $[R_4]$ are connected and $\mathcal{I}_4 = \{\{S_1\}, \{S_2, R_1, R_2, R_3, R_4\}, \{S_3\}\}$ is obtained from \mathcal{I}_3 by combining two connected equivalent classes. In \mathcal{I}_4 , again by (1) of Definition 4.9, $[S_2]$ and $[S_1]$ are connected and $\mathcal{I}_5 = \{\{S_1, S_2, R_1, R_2, R_3, R_4\}, \{S_3\}\}$ is obtained from \mathcal{I}_4 by combining two connected equivalent classes. Thus, $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_5$ satisfy the conditions of Definition 4.11. So $\mathcal{I}_c = \mathcal{I}_5$ is a character partition of Π . Since $[S_1] = [S_2]$, then \mathcal{I}_c is not compatible.

Similarly, for the region graph in Fig. 4 (b), we can find that $\mathcal{I}_c = \{\{S_1, P_1, P_2, P_3, P_4\}, \{S_2\}, \{S_3\}\}$ is a character partition of Π . Since $\Lambda_2 \subseteq [S_1]_1$, then \mathcal{I}_c is not compatible.

Lemma 4.13: Let \mathcal{I} be a partition of Π . If \mathcal{I} is compatible, then $\text{RG}(D^{**})$ is feasible.

Proof: The proof is given in Appendix A. ■

Lemma 4.14: Suppose $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfies the conditions of Lemma 4.6 and \mathcal{I}_c is a character partition of Π . For any $[R] \in \mathcal{I}_c$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, if $Q \in [R]_{i_1, i_2}$ and $d_Q \neq 0$, then $d_{Q'} \in \langle d_Q \rangle, \forall Q' \in [R]_{i_1, i_2}$.

Proof: The proof is given in Appendix B. ■

Theorem 4.15: Let $\text{RG}(D^{**})$ be terminal-separable and \mathcal{I}_c be a character partition of Π . Then $\text{RG}(D^{**})$ is feasible if and

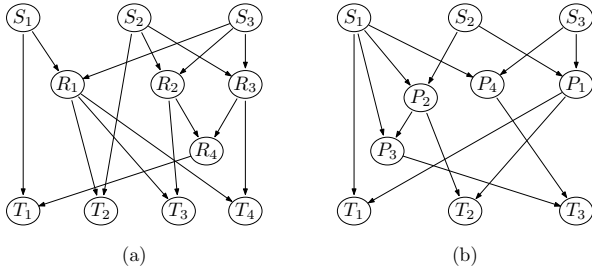


Fig 4. Examples of region graph.

only if \mathcal{I}_c is compatible. Moreover, it is $\{|\Pi|, n\}$ -polynomial time complexity to determine feasibility of $\text{RG}(D^{**})$.

Proof: If \mathcal{I}_c is compatible, then by Lemma 4.13, $\text{RG}(D^{**})$ is feasible. Conversely, suppose $\text{RG}(D^{**})$ is feasible and $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfies the conditions of Lemma 4.6. We shall prove that \mathcal{I}_c is compatible.

For any $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, if $[S_{i_1}] = [S_{i_2}]$, then $S_{i_1} \in [S_{i_2}]_{i_1, i_2}$. By Definition 3.3 and Lemma 4.14, $d_{S_{i_1}} = \alpha_{i_1} \in \langle d_{S_{i_2}} \rangle = \langle \alpha_{i_2} \rangle$, a contradiction. So $[S_{i_1}] \neq [S_{i_2}]$. Thus, by proper naming, we can assume $\mathcal{I}_c = \{[S_1], [S_2], [S_3], [R_4], \dots, [R_K]\}$. Moreover, by Definition 4.11, no pair of equivalent classes of \mathcal{I}_c are connected.

For any $j \in [n]$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, suppose $\Lambda_j \subseteq [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup (\cup_{\ell=4}^K [R_\ell]_{i_1, i_2})$. Then by Lemma 4.14 and Equation (4), we have

$$d_Q \in \langle d_{S_{i_1}} \rangle = \langle \alpha_{i_1} \rangle, \forall Q \in [S_{i_1}]_{i_1} \quad (5)$$

and

$$d_Q \in \langle d_{S_{i_2}} \rangle = \langle \alpha_{i_2} \rangle, \forall Q \in [S_{i_1}]_{i_2}. \quad (6)$$

Note that \tilde{C}_Π satisfies condition (2) of Lemma 4.6. By Equation (5), (6) and Definition 3.7, we can easily see that $d_R \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle$ for all $R \in [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup (\cup_{\ell=4}^K [R_\ell]_{i_1, i_2})$. Then $d_R \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle$ for all $R \in \Lambda_j$ and $\tilde{\alpha} \notin \langle d_R; R \in \Lambda_j \rangle$, which contradicts to the assumption that \tilde{C}_Π satisfies condition (3) of Lemma 4.6. Thus, $\Lambda_j \not\subseteq [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup (\cup_{\lambda=4}^K [R_\lambda]_{i_1, i_2})$. By Definition 4.10, \mathcal{I}_c is compatible.

By Definition 4.11, the following algorithm output a character partition of Π .

Algorithm 2: Partitioning algorithm (Π, S) :

```

L = 0;
While there are  $R', R'' \in \mathcal{I}_L$  which are  $S$ -connected do
  Let  $\mathcal{I}_{L+1}$  be a contraction of  $\mathcal{I}_L$  by combining  $R'$  and  $R''$ ;
  If  $[S_i] = [S_j]$  for some  $\{i, j\} \subseteq \{1, 2, 3\}$  then
     $\mathcal{I}_c = \mathcal{I}_L$ ;
    return  $\mathcal{I}_c$ ;
  stop;
else
  L = L + 1;
 $\mathcal{I}_c = \mathcal{I}_L$ ;
return  $\mathcal{I}_c$ ;

```

Clearly, there are at most $|\Pi|$ rounds in Algorithm 2 before output \mathcal{I}_c . In each round, we need to determine whether there

are two S -connected equivalent classes, which can be done in time $O(|S|) = O(n)$ by Definition 4.9. Thus, it is $\{|\Pi|, n\}$ -polynomial time complexity to determine whether $\text{RG}(D^{**})$ is feasible. ■

Consider the region graph in Fig. 3. We can check that the partition \mathcal{I} in Example 4.8 is a character partition of Π . Since \mathcal{I} is compatible, so the region graph is feasible. Let $\mathbb{F} = GF(p)$ for a sufficiently large prime p . Let $d_{R_1} = d_{R_4} = d_{R_7} = \alpha_1$, $d_{R_2} = 2\alpha_1 + 3\alpha_3$, $d_{R_3} = d_{R_5} = \alpha_2 + \alpha_3$, $d_{R_6} = \alpha_1 + 3\alpha_2$. Then $\{d_R; R \in \Pi\}$ is a linear solution of the graph.

Similar to the information flow decomposition technique used in [13], we can reduce any compatible partition of Π into a minimal compatible partition \mathcal{I}_m , i.e., \mathcal{I}_m is a compatible partition of Π but any contraction of \mathcal{I}_m is not compatible. Then we can construct an optimal linear solution of $\text{RG}(D^{**})$ on \mathcal{I}_m using the method in the proof of Lemma 4.13.

For the two region graphs in Fig. 4, we have seen that there is a character partition of Π that is not compatible. So by Theorem 4.15, these two region graphs are not feasible.

In [6], a necessary and sufficient condition for solvability of a 3s/3t sum-network was given based on a set of connection conditions. By our method, we can give another sufficient and necessary condition for solvability of 3s/3t sum-networks which is different from [6]:

Theorem 4.16: Suppose $\text{RG}(D^{**})$ has three terminal regions. Then $\text{RG}(D^{**})$ is not feasible if and only if it is terminal separable and the following condition (C-IR) hold:

(C-IR) By proper naming, there is a $P_1 \in \text{reg}^\circ(S_2, S_3)$ and a $P_2 \in \text{reg}^\circ(S_1, S_2)$ such that $\Lambda_1 = \{S_1, P_1\}$, $\Lambda_2 = \{P_1, P_2\}$ and $\Lambda_3 \subseteq \text{reg}(S_1, P_2) \cup \text{reg}(S_1, S_3)$.

Proof: The proof is given in Appendix C. ■

Fig. 4 (b) is an illustration of infeasible region graph of 3s/3t sum-network.

V. CONCLUSIONS AND DISCUSSIONS

We investigated the network coding problem of a special subclass of 3s/nt sum-networks termed as terminal-separable networks using a network region decomposition method. We give a necessary and sufficient condition for solvability of terminal separable networks as well as a simple characterization of solvability of 3s/3t sum-networks. The region decomposition method is shown to be an efficient tool for analyzing the structure of a network and helps to investigate the network coding problem of a communication network. By more intensive analysis, we can also give a characterization of solvability of 3s/4t sum-networks, which is our future work.

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APPENDIX A PROOF OF LEMMA 4.13

Here, we prove Lemma 4.13. First, we prove two lemmas.

Lemma A.1: Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2 + \alpha_3\}$, $\mathcal{B}_2 = \{\alpha_2, \alpha_1 + \alpha_3\}$, $\mathcal{B}_3 = \{\alpha_3, \alpha_1 + \alpha_2\}$ and $K \geq 3$ is an integer. If \mathbb{F} is sufficiently large, then there are $K - 3$ subsets $\mathcal{B}_4 = \{\beta_{1,2}^{(4)}, \beta_{1,3}^{(4)}, \beta_{2,3}^{(4)}\}, \dots, \mathcal{B}_K = \{\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}, \beta_{2,3}^{(K)}\} \subseteq \mathbb{F}^3$ such that $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \dots, \mathcal{B}_K\}$ satisfies the following conditions:

- (1) For any $\ell \in \{4, \dots, K\}$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, $\beta_{i_1, i_2}^{(\ell)} \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle$;
- (2) For any $\ell \in \{1, \dots, K\}$ and $\{\gamma, \gamma'\} \subseteq \mathcal{B}_\ell$, $\bar{\alpha} \in \langle \gamma, \gamma' \rangle$;
- (3) If $\{\gamma, \gamma', \gamma''\} \subseteq \cup_{\ell=1}^K \mathcal{B}_\ell$ such that $\{\gamma, \gamma', \gamma''\} \not\subseteq \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, and $\{\gamma, \gamma', \gamma''\} \neq \{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\}, \forall \ell \in \{4, \dots, K\}$, then γ, γ' and γ'' are linearly independent;
- (4) For any pair $\{\gamma, \gamma'\} \subseteq \cup_{\ell=1}^K \mathcal{B}_\ell$, γ and γ' are linearly independent.

Proof: We can prove this lemma by induction.

Clearly, when $K = 3$, the collection $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ satisfies conditions (1)–(4).

Now suppose $K > 3$ and there is a collection $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}\}$ which satisfies conditions (1)–(4). We want to construct a subset $\mathcal{B}_K = \{\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}, \beta_{2,3}^{(K)}\} \subseteq \mathbb{F}^3$ such that the collection $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies conditions (1)–(4). The subset \mathcal{B}_K can be constructed as follows:

Let Φ_{K-1} be the set of all pairs $\{\gamma, \gamma'\} \subseteq \cup_{\ell=1}^{K-1} \mathcal{B}_\ell$ such that $\{\gamma, \gamma'\} \not\subseteq \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then $\langle \gamma, \gamma' \rangle \cap \langle \alpha_{i_1}, \alpha_{i_2} \rangle$ is an 1-dimensional subspace of \mathbb{F}^3 . Let $\langle \gamma, \gamma' \rangle_{i_1, i_2}$ be a fixed non-zero vector in $\langle \gamma, \gamma' \rangle \cap \langle \alpha_{i_1}, \alpha_{i_2} \rangle$. Let

$$\Psi_{K-1} = \bigcup_{\{\gamma, \gamma'\} \in \Phi_{K-1}} \{\langle \gamma, \gamma' \rangle_{1,2}, \langle \gamma, \gamma' \rangle_{1,3}, \langle \gamma, \gamma' \rangle_{2,3}\}.$$

Since \mathbb{F} is sufficiently large, then there exists a $\beta^{(K)} \in \mathbb{F}^3$ such that

$$\beta^{(K)} \notin \langle \bar{\alpha}, \gamma \rangle, \forall \gamma \in \Psi_{K-1}. \quad (7)$$

For each $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, let

$$0 \neq \beta_{i_1, i_2}^{(K)} \in \langle \beta^{(K)}, \bar{\alpha} \rangle \cap \langle \alpha_{i_1}, \alpha_{i_2} \rangle \quad (8)$$

where 0 is the zero vector of \mathbb{F}^3 . Let $\mathcal{B}_K = \{\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}, \beta_{2,3}^{(K)}\}$. We shall prove that the collection $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies conditions (1)–(4).

By Equation (8), we have $\beta_{i_1, i_2}^{(K)} \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$. So $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies condition (1).

By assumption, $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}\}$ satisfies condition (2), then for any $\ell \in \{1, \dots, K-1\}$ and $\{\gamma, \gamma'\} \subseteq \mathcal{B}_\ell$, the pair $\{\gamma, \gamma'\}$ is in Φ_{K-1} . Moreover, since $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}\}$ satisfies condition (1), then $\{\gamma, \gamma'\} \subseteq \mathcal{B}_\ell \subseteq \langle \alpha_1, \alpha_2 \rangle \cup \langle \alpha_1, \alpha_3 \rangle \cup \langle \alpha_2, \alpha_3 \rangle$. So $\{\gamma, \gamma'\} \subseteq \{\langle \gamma, \gamma' \rangle_{1,2}, \langle \gamma, \gamma' \rangle_{1,3}, \langle \gamma, \gamma' \rangle_{2,3}\}$. Thus, we have $\cup_{\ell=1}^{K-1} \mathcal{B}_\ell \subseteq \Psi_{K-1}$. By Equation (7), for any $\gamma \in \cup_{\ell=1}^{K-1} \mathcal{B}_\ell$,

$$\gamma \notin \langle \beta^{(K)}, \bar{\alpha} \rangle. \quad (9)$$

In particular, we have $\alpha_j \notin \langle \beta^{(K)}, \bar{\alpha} \rangle, j = 1, 2, 3$. So by Equation (8), $\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}$ and $\beta_{2,3}^{(K)}$ are mutually linearly independent and $\bar{\alpha} \in \langle \gamma, \gamma' \rangle, \forall \{\gamma, \gamma'\} \subseteq \mathcal{B}_K$. Thus, $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies condition (2).

Now, we prove that $\cup_{\ell=1}^K \mathcal{B}_\ell$ satisfies condition (3). Suppose $\{\gamma, \gamma', \gamma''\} \subseteq \cup_{\ell=1}^K \mathcal{B}_\ell$ such that $\{\gamma, \gamma', \gamma''\} \not\subseteq \langle \alpha_{i_1}, \alpha_{i_2} \rangle$ for any $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ and $\{\gamma, \gamma', \gamma''\} \neq \{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\}$ for any $\ell \in \{1, \dots, K\}$. We have the following three cases:

Case 1: $\{\gamma, \gamma', \gamma''\} \subseteq \cup_{\ell=1}^{K-1} \mathcal{B}_\ell$. By the induction assumption, γ, γ' and γ'' are linearly independent.

Case 2: $\{\gamma, \gamma'\} \subseteq \cup_{\ell=1}^{K-1} \mathcal{B}_\ell$ and $\gamma'' \in \mathcal{B}_K$. We have the following two subcases:

Case 2.1: $\{\gamma, \gamma'\} \subseteq \langle \alpha_{\ell_1}, \alpha_{\ell_2} \rangle$ for some $\{\ell_1, \ell_2\} \subseteq \{1, 2, 3\}$. By assumption of $\{\gamma, \gamma', \gamma''\}$, we have $\gamma'' \notin \langle \alpha_{\ell_1}, \alpha_{\ell_2} \rangle$. So γ, γ' and γ'' are linearly independent.

Case 2.2: $\{\gamma, \gamma'\} \not\subseteq \langle \alpha_{\ell_1}, \alpha_{\ell_2} \rangle, \forall \{\ell_1, \ell_2\} \subseteq \{1, 2, 3\}$. Then the pair $\{\gamma, \gamma'\}$ is in the set Φ_{K-1} . So we have $\gamma'' \notin \langle \gamma, \gamma' \rangle$. (Otherwise, $\gamma'' \in \{\langle \gamma, \gamma' \rangle_{1,2}, \langle \gamma, \gamma' \rangle_{1,3}, \langle \gamma, \gamma' \rangle_{2,3}\} \subseteq \Psi_{K-1}$ and by Equation (8), $\beta^{(K)} \in \langle \bar{\alpha}, \gamma'' \rangle$, which contradicts to Equation (7).) Thus, γ, γ' and γ'' are linearly independent.

Case 3: $\gamma \in \cup_{\ell=1}^{K-1} \mathcal{B}_\ell$ and $\{\gamma', \gamma''\} \subseteq \mathcal{B}_K$. By Equations (8) and (9), $\gamma \notin \langle \beta^{(K)}, \bar{\alpha} \rangle = \langle \gamma', \gamma'' \rangle$. So γ, γ' and γ'' are linearly independent.

Thus, $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies conditions (3).

Clearly, if $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies conditions (3), then for any $\{\gamma, \gamma'\} \subseteq \cup_{\ell=1}^K \mathcal{B}_\ell$, we can find a $\gamma'' \in \cup_{\ell=1}^K \mathcal{B}_\ell$ such that γ, γ' and γ'' are linearly independent. So γ and γ' are linearly independent and $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ satisfies conditions (4).

By induction, for all $K \geq 3$, we can always find a collection $\{\mathcal{B}_1, \dots, \mathcal{B}_{K-1}, \mathcal{B}_K\}$ which satisfies conditions (1)–(4). ■

We give an example of Lemma A.1 in the below. To simplify our discussion, we assume that $\mathbb{F} = GF(p)$, where p is a sufficiently large prime.

Example A.2: According to Lemma A.1, $\mathcal{B}_1 = \{\alpha_1, \alpha_2 + \alpha_3\}$, $\mathcal{B}_2 = \{\alpha_2, \alpha_1 + \alpha_3\}$, $\mathcal{B}_3 = \{\alpha_3, \alpha_1 + \alpha_2\}$. Then $\Phi_3 = \{\{\alpha_1, \alpha_2 + \alpha_3\}, \{\alpha_2, \alpha_1 + \alpha_3\}, \{\alpha_3, \alpha_1 + \alpha_2\}, \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_3\}, \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2\}, \{\alpha_1 + \alpha_3, \alpha_1 + \alpha_2\}\}$ and $\Psi_3 = \{\alpha_1, \alpha_2 + \alpha_3\} \cup \{\alpha_2, \alpha_1 + \alpha_3\} \cup \{\alpha_3, \alpha_1 + \alpha_2\} \cup \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 - \alpha_2\} \cup \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1 - \alpha_3\} \cup \{\alpha_1 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 - \alpha_3\}$. We can check that $\alpha_1 + 3\alpha_2 \notin \langle \bar{\alpha}, \gamma \rangle, \forall \gamma \in \Psi_3$. Let $\beta^{(4)} = \alpha_1 + 3\alpha_2$ and $\mathcal{B}_4 = \{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_3, 2\alpha_2 - \alpha_3\}$. Then the collection $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$ satisfies conditions (1)–(4) of Lemma A.1.

Similarly, we can construct a subset $\mathcal{B}_5 = \{2\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_3, \alpha_2 - 2\alpha_3\}$ such that the collection $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\}$ satisfies conditions (1)–(4) of Lemma A.1.

Lemma A.3: Let $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ and $\{\Delta_1, \dots, \Delta_K\}$ be a partition of $\text{reg}(S_{i_1}, S_{i_2})$ such that $\text{reg}(\Delta_i) = \Delta_i, i = 1, \dots, K$. Let $\tilde{C}_{i_1, i_2} = \{d_R; R \in \text{reg}(S_{i_1}, S_{i_2})\} \subseteq \langle \alpha_{i_1}, \alpha_{i_2} \rangle$ be such that:

- (1) If $\{R, R'\} \subseteq \Delta_i$ for some $i \in [K]$, then $d_R = d_{R'}$;
- (2) If $\{R, R'\} \not\subseteq \Delta_i$ for any $i \in [K]$, then d_R and $d_{R'}$ are linearly independent.

Then $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle, \forall R \in \text{reg}^\circ(S_{i_1}, S_{i_2})$.

Proof: Suppose $R \in \text{reg}^\circ(S_{i_1}, S_{i_2})$. Then by Definition 3.7, $\text{In}(R) \subseteq \text{reg}(S_{i_1}, S_{i_2})$. We have the following two cases:

Case 1: $\text{In}(R) \subseteq \Delta_i$ for some $i \in \{1, \dots, K\}$. Then by Definition 3.7, $R \in \text{reg}(\Delta_i)$. Since by the assumption of this lemma, $\text{reg}(\Delta_i) = \Delta_i$, then $R \in \Delta_i$ and, by condition (1), $d_R = d_{R'}$ for all $R' \in \text{In}(R)$. Thus, $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle$.

Case 2: $\text{In}(R) \not\subseteq \Delta_i$ for any $i \in \{1, \dots, K\}$. By Definition 3.4, each non-source region has at least two parents. Since $\{\Delta_1, \dots, \Delta_K\}$ is a partition of $\text{reg}(S_{i_1}, S_{i_2})$, then there exists a subset $\{i_1, i_2\} \subseteq \{1, \dots, K\}$ such that $\text{In}(R) \cap \Delta_{i_1} \neq \emptyset$ and $\text{In}(R) \cap \Delta_{i_2} \neq \emptyset$. Assume $R'_1 \in \text{In}(R) \cap \Delta_{i_1}$ and $R'_2 \in \text{In}(R) \cap \Delta_{i_2}$. Then by condition (2), $d_{R'_1}$ and $d_{R'_2}$ are linearly independent and $d_R \in \langle d_{R'_1}, d_{R'_2} \rangle = \langle \alpha_1, \alpha_2 \rangle$. So $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle$. ■

Definition A.4: Let $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ be a partition of Π . A subset $\{Q, Q', Q''\} \subseteq \Pi$ is called an \mathcal{I} -independent set if the following three conditions hold:

- (1) $|\{Q, Q', Q''\} \cap [[R]]| \leq 1$ for any equivalent class $[R] \in \mathcal{I}$ and any subclass $[[R]]$ of $[R]$;
- (2) $\{Q, Q', Q''\} \not\subseteq [R]$ for any equivalent class $R \in \mathcal{I}$;
- (3) $\{Q, Q', Q''\} \not\subseteq [S_i]_i \cup [S_j]_j \cup (\cup_{\ell=4}^K [R_\ell]_{i,j})$ for any pair $\{i, j\} \subseteq \{1, 2, 3\}$.

Now we can prove Lemma 4.13

Proof of Lemma 4.13: Since \mathcal{I} is compatible, by Definition 4.10, we can assume $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$. Let $\mathcal{B}_1, \dots, \mathcal{B}_K$ be as in Lemma A.1. We construct a code $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ as follows:

- For $j \in \{1, 2, 3\}$ and $R \in [S_j]_j$, let $d_R = \alpha_j$;
- For $j \in \{1, 2, 3\}$ and $R \in [S_j]_{i_1, i_2}$, let $d_R = \alpha_{i_1} + \alpha_{i_2}$, where $\{i_1, i_2\} = \{1, 2, 3\} \setminus \{j\}$;
- For $j \in \{4, \dots, K\}$, $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ and $R \in [R_j]_{i_1, i_2}$, let $d_R = \beta_{i_1, i_2}^{(j)}$.

We shall prove that $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfies the conditions of Lemma 4.6.

By the construction of \tilde{C}_Π , we have $d_{S_j} = \alpha_j, j = 1, 2, 3$. Moreover, since \mathcal{I} is compatible, then for each $[R_\ell] \in \mathcal{I}$ and $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, we have $[R_\ell]_{i_1, i_2} = \text{reg}([R_\ell]_{i_1, i_2})$. (Otherwise, by Definition 3.7, there is an $R \in \text{reg}([R_\ell]_{i_1, i_2}) \setminus [R_\ell]_{i_1, i_2}$. By condition (2) of Definition 4.9, $[R_\ell]$ and $[R]$ are connected, which contradicts to the assumption that \mathcal{I} is compatible.) Now, let $\Delta_i = [R_i]_{i_1, i_2}, i = 1, \dots, K$, where $[R_i] = [S_i], i = 1, 2, 3$. By the construction, $\tilde{C}_{i_1, i_2} = \{d_R; R \in \text{reg}(S_{i_1}, S_{i_2})\}$ satisfies the conditions of Lemma A.3. So $d_R \in \langle d_{R'}; R' \in \text{In}(R) \rangle, \forall R \in \text{reg}^\circ(S_{i_1}, S_{i_2})$.

Finally, we prove that \tilde{C}_Π satisfies condition (3) of Lemma 4.6. For each $\Lambda_j, j \in [n]$, we have the following two cases:

Case 1: There is an $[R_\ell] \in \mathcal{I}$ such that Λ_j intersects with at least two subclasses of $[R_\ell]$. Suppose $Q_1 \in \Lambda_j \cap [[R_\ell]]_1$ and $Q_2 \in \Lambda_j \cap [[R_\ell]]_2$, where $[[R_\ell]]_1$ and $[[R_\ell]]_2$ are two different subclasses of $[R_\ell]$. Then by the construction of \tilde{C}_Π , $\{d_{Q_1}, d_{Q_2}\} \subseteq \mathcal{B}_\ell$ and $\bar{\alpha} \in \langle d_{Q_1}, d_{Q_2} \rangle$.

Case 2: For each $[R_\ell] \in \mathcal{I}$, Θ_j intersects with at most one subclass of $[R_\ell]$. Since \mathcal{I} is compatible, then we can always find a subset $\{Q_1, Q_2, Q_3\} \subseteq \Lambda_j$ such that $\{Q_1, Q_2, Q_3\}$ is an \mathcal{I} -independent set. By the construction of \tilde{C}_Π , $\{d_{Q_1}, d_{Q_2}, d_{Q_3}\} \not\subseteq \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, and $\{d_{Q_1}, d_{Q_2}, d_{Q_3}\} \neq \{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\}, \forall \ell \in \{4, \dots, K\}$. So d_{Q_1}, d_{Q_2} and d_{Q_3} are linearly independent. Thus, $\bar{\alpha} \in \langle d_{Q_1}, d_{Q_2}, d_{Q_3} \rangle = \mathbb{F}^3$.

By the above discussion, \tilde{C}_Π satisfies the conditions of Lemma 4.6. So $\text{RG}(D^{**})$ is feasible. ■

Here, we make an example to illustrate the construction of \tilde{C}_Π in the proof of Lemma 4.13.

APPENDIX B PROOF OF LEMMA 4.14

Here, we prove Lemma 4.14.

Suppose $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ is a collection that satisfies the conditions of Lemma 4.6. Note that $\text{RG}(D^{**})$ is acyclic and $d_{S_i} = \alpha_i \neq 0, i = 1, 2, 3$. If there is an $R \in \Pi$ such that $d_R = 0$, then we can always find an $R_0 \in \Pi$ such that $d_{R_0} = 0$ and $d_{R'} \neq 0, \forall R' \in \text{In}(R_0)$. We redefine d_{R_0} by letting $d_{R_0} = d_{R'}$ for a fixed $R' \in \text{In}(R_0)$. Then the resulted code $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ still satisfies the conditions of Lemma 4.6 and $d_{R_0} \neq 0$. We can perform this operation continuously until $d_R \neq 0$ for all $R \in \Pi$ and the resulted code $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ still satisfies the conditions of Lemma 4.6. So we can assume, without loss of generality, that $d_R \neq 0$ for all $R \in \Pi$.

To prove Lemma 4.14, the key is to prove that all equivalent class $[R] \in \mathcal{I}_c$ satisfies the following property:

- *Property (a):* For any pair $\{Q, Q'\} \subseteq [R]$, $d_{Q'} \in \langle d_Q, \bar{\alpha} \rangle$.

To prove this, we first prove the following two lemmas.

Lemma B.1: Let \mathcal{I} be a partition of Π and $[R] \in \mathcal{I}$ satisfies Property (a). Then for any $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ and any pair $\{Q, Q'\} \subseteq [R]_{i_1, i_2}$, $d_{Q'} \in \langle d_Q \rangle$. Moreover, for any subclass $[[R]]$ of $[R]$ and any pair $\{Q, Q'\} \subseteq [[R]]$, $d_{Q'} \in \langle d_Q \rangle$.

Proof: Since \tilde{C}_Π satisfies conditions (1) and (2) of Lemma 4.6, then by Definition 3.7, we have $d_W \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \forall W \in \text{reg}(S_{i_1}, S_{i_2})$. By assumption and Equation

(3), $\{Q, Q'\} \subseteq [R]_{i_1, i_2} \subseteq \text{reg}(S_{i_1}, S_{i_2})$. So $d_Q, d_{Q'} \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle$. Meanwhile, since $[R] \in \mathcal{I}$ satisfies Property (a), then $d_{Q'} \in \langle d_Q, \bar{\alpha} \rangle$. So $d_{Q'} \in \langle \alpha_{i_1}, \alpha_{i_2} \rangle \cap \langle d_Q, \bar{\alpha} \rangle = \langle d_Q \rangle$ and the first claim is true.

We now prove the second claim. If $[R] \neq [S_i], \forall i \in \{1, 2, 3\}$, then by Definition 4.7, $[[R]] = [R]_{i_1, i_2}$ for some $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ and by the proven result, $d_{Q'} \in \langle d_Q \rangle$. If $[R] = [S_i]$ for some $i \in \{1, 2, 3\}$, then by Definition 4.7, we have the following two cases:

Case 1: $[[R]] = [S_i]_i = [S_i]_{i, j_1} \cup [S_i]_{i, j_2}$, where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$. By the proven result, we have $\alpha_i = d_{S_i} \in \langle d_Q \rangle$ and $d_{Q'} \in \langle d_{S_i} \rangle$. So $d_{Q'} \in \langle d_Q \rangle$.

Case 2: $[[R]] = [S_i]_{j_1, j_2}$, where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$. By the proven result, we have $d_{Q'} \in \langle d_Q \rangle$.

In both cases, we have $d_{Q'} \in \langle d_Q \rangle$. So the second claim is true. ■

Lemma B.2: Suppose $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ is a partition of Π in which all equivalent classes satisfy Property (a). Suppose $\{[R], [R']\} \subseteq \mathcal{I}$ and there is a Λ_j such that $\Lambda_j \subseteq [[R']] \cup [[R'']]$, where $[[R']]$ (resp. $[[R'']]$) is a subclass of $[R']$ (resp. $[R'']$). Then $\bar{\alpha} \in \langle d_{P'}, d_{P''} \rangle$ for any $P' \in \Lambda_j \cap [[R']]$ and $P'' \in \Lambda_j \cap [[R'']]$.

Proof: Since \tilde{C}_Π satisfies condition (3) of Lemma 4.6 and $\Lambda_j \subseteq [[R']] \cup [[R'']]$, then $\bar{\alpha} \in \langle d_R; R \in \Lambda_j \rangle = \langle d_R; R \in (\Lambda_j \cap [[R']]) \cup (\Lambda_j \cap [[R'']]) \rangle$. By Lemma B.1, $\langle d_R; R \in (\Lambda_j \cap [[R']]) \cup (\Lambda_j \cap [[R'']]) \rangle = \langle d_{P'}, d_{P''} \rangle$. So $\bar{\alpha} \in \langle d_{P'}, d_{P''} \rangle$. ■

Lemma B.3: Suppose $\mathcal{I} = \{[S_1], [S_2], [S_3], \dots, [R_K]\}$ is a partition of Π and \mathcal{I}' is a contraction of \mathcal{I} by combining two connected equivalent classes $[R']$ and $[R'']$ in \mathcal{I} . If all equivalent classes in \mathcal{I} satisfy Property (a), then all equivalent classes in \mathcal{I}' satisfy Property (a).

Proof: Suppose $[R] \in \mathcal{I}'$. If $[R] \neq [R'] \cup [R'']$, then $[R] \in \mathcal{I}$, and by assumption, $[R]$ satisfies property (a). Now we suppose $[R] = [R'] \cup [R'']$. Since, $[R']$ and $[R'']$ are connected, by Definition 4.9, we have the following two cases:

Case 1: There is a $\Lambda_j \subseteq [[R']] \cup [[R'']]$, where $[[R']]$ (resp. $[[R'']]$) is a subclass of $[R']$ (resp. $[R'']$). By Lemma B.2, $\bar{\alpha} \in \langle d_{P'}, d_{P''} \rangle$, where $P' \in \Lambda_j \cap [[R']]$ and $P'' \in \Lambda_j \cap [[R'']]$. Then $d_{P''} \in \langle d_{P'}, \bar{\alpha} \rangle$ and $d_{P'} \in \langle d_{P''}, \bar{\alpha} \rangle$. Since, by assumption, $[R']$ and $[R'']$ satisfy property (a), then $\langle d_Q, \bar{\alpha} \rangle = \langle d_{P'}, \bar{\alpha} \rangle = \langle d_{P''}, \bar{\alpha} \rangle$ and $d_{Q'} \in \langle d_Q, \bar{\alpha} \rangle, \forall \{Q, Q'\} \subseteq [R] = [R'] \cup [R'']$.

Case 2: There is a subset $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ such that $\text{reg}([R']_{i_1, i_2}) \cap \text{reg}([R'']_{i_1, i_2}) \neq \emptyset$. Pick a $Q_0 \in \text{reg}([R']_{i_1, i_2}) \cap \text{reg}([R'']_{i_1, i_2})$. Since, by assumption, $[R']$ and $[R'']$ satisfy property (a), then $\langle d_Q, \bar{\alpha} \rangle = \langle d_{Q_0}, \bar{\alpha} \rangle = \langle d_{Q'}, \bar{\alpha} \rangle$ and $d_{Q'} \in \langle d_Q, \bar{\alpha} \rangle, \forall \{Q, Q'\} \subseteq [R] = [R'] \cup [R'']$.

In both cases, $[R] = [R'] \cup [R'']$ satisfies property (a). Thus, all equivalent classes in \mathcal{I}' satisfy Property (a). ■

Now we can prove Lemma 4.14.

Proof of Lemma 4.14: Since each equivalent class $[R]$ in \mathcal{I}_0 contains exactly one region R , so $[R]$ naturally satisfies property (a) and $[S_i] \neq [S_j]$ for all $\{i, j\} \subseteq \{1, 2, 3\}$.

By Definition 4.11, $\mathcal{I}_c = \mathcal{I}_L$, where $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_L = \mathcal{I}_c$ is a sequence of partitions of Π such that \mathcal{I}_ℓ is a contraction of $\mathcal{I}_{\ell-1}$ by combining two connected equivalent classes in $\mathcal{I}_{\ell-1}$ and, for any $\{i, j\} \subseteq \{1, 2, 3\}$, $[S_i] \neq [S_j]$ in $\mathcal{I}_{\ell-1}$,

$\ell = 1, \dots, L$. So by Lemma B.3, all equivalent classes in \mathcal{I}_ℓ satisfy Property (a). In particular, all equivalent classes in $\mathcal{I}_c = \mathcal{I}_L$ satisfies property (a). Then the conclusion of Lemma 4.14 is obtained by Lemma B.1. ■

APPENDIX C

PROOF OF THEOREM 4.16

Here, we prove Theorem 4.16. First, we prove some lemmas.

Lemma C.1: If $\text{RG}(D^{**})$ has two terminal regions, then $\text{RG}(D^{**})$ is feasible.

Proof: Suppose $\text{RG}(D^{**})$ has two terminal regions T_1 and T_2 . We have the following two cases:

Case 1: $\Omega_{1,2} \neq \emptyset$. Then there is a $Q \in D^{**} \setminus \Pi$ such that $Q \rightarrow T_i, i = 1, 2$. Similar to what we did in Remark 4.4, we can first construct a code on the set $\{R \in D^{**}; R \rightarrow Q\}$ such that $d_Q = \bar{\alpha}$. Then for all R such that $Q \rightarrow R \rightarrow T_i$ for some $i \in \{1, 2\}$, let $d_R = \bar{\alpha}$. By this construction, we obtain a solution of $\text{RG}(D^{**})$. So $\text{RG}(D^{**})$ is feasible.

Case 2: $\Omega_{1,2} = \emptyset$. Then $\text{RG}(D^{**})$ is terminal separable. From Lemma 4.5, we have $\Lambda_j \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, and $|\Lambda_j| \geq 2, j = 1, 2$. By enumerating, we have the following three subcases:

Case 2.1: $|\Lambda_1| > 2$ and $|\Lambda_2| > 2$. Let $\mathcal{I}_0 = \{[R]; R \in \Pi\}$, where $[R] = \{R\}$ for all $R \in \Pi$. Clearly, \mathcal{I}_0 is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.2: $|\Lambda_1| > 2$ and $|\Lambda_2| = 2$. Let $\mathcal{I} = \{\Lambda_2\} \cup \{[R]; R \in \Pi \setminus \Lambda_2\}$, where $[R] = \{R\}$ for all $R \in \Pi \setminus \Lambda_2$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.3: $|\Lambda_1| = |\Lambda_2| = 2$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let $\mathcal{I} = \{\Lambda_1, \Lambda_2\} \cup \{[R]; R \in \Pi \setminus (\Lambda_1 \cup \Lambda_2)\}$, where $[R] = \{R\}$ for all $R \in \Pi \setminus (\Lambda_1 \cup \Lambda_2)$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.4: $|\Lambda_1| = |\Lambda_2| = 2$ and $\Lambda_1 \cap \Lambda_2 \neq \emptyset$. If $\Lambda_1 = \Lambda_2$, then it is easy to construct a code \tilde{C}_Π satisfies the conditions of Lemma 4.6. So $\text{RG}(D^{**})$ is feasible. Thus, we can assume $\Lambda_1 \neq \Lambda_2$. Then by proper naming, we can assume $\Lambda_1 = \{Q_1, Q_2\}$ and $\Lambda_2 = \{Q_1, Q_3\}$. By Lemma 4.5, $\{Q_1, Q_2\} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$ and $\{Q_1, Q_3\} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$ for all $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then one of the following two cases hold:

Case 2.4.1: $\{Q_2, Q_3\} \subseteq \text{reg}(S_{i_1}, S_{i_2})$ for some $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. Let $\mathcal{I} = \{[Q_1]\} \cup \{[R]; R \in \Pi \setminus [Q_1]\}$, where $[Q_1] = \{Q_1\} \cup \text{reg}(Q_2, Q_3)$ and $[R] = \{R\}$ for all $R \in \Pi \setminus [Q_1]$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.4.2: $\{Q_2, Q_3\} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$ for all $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. Let $\mathcal{I} = \{[Q_1]\} \cup \{[R]; R \in \Pi \setminus [Q_1]\}$, where $[Q_1] = \{Q_1, Q_2, Q_3\}$ and $[R] = \{R\}$ for all $R \in \Pi \setminus [Q_1]$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

By the above discussions, we proved that $\text{RG}(D^{**})$ is feasible. ■

Lemma C.2: Suppose $\text{RG}(D^{**})$ has three terminal regions and is terminal separable. Then $\text{RG}(D^{**})$ is feasible if one of the following conditions hold:

- (1) $|\Lambda_{j_1}| \geq 3$ and $|\Lambda_{j_2}| \geq 3$ for some $\{j_1, j_2\} \subseteq \{1, 2, 3\}$;
- (2) For any $\{j_1, j_2\} \subseteq \{1, 2, 3\}$, if $|\Lambda_{j_1}| = |\Lambda_{j_2}| = 2$, then $\Lambda_{j_1} \cap \Lambda_{j_2} = \emptyset$;
- (3) $S_i \notin \Lambda_j$ for all $i, j \in \{1, 2, 3\}$;
- (4) There is a subset $\{\ell', \ell''\} \subseteq \{1, 2, 3\}$ such that $\Lambda_j \cap \text{reg}^\circ(S_{\ell'}, S_{\ell''}) \neq \emptyset$ for all $j \in \{1, 2, 3\}$.

Proof: 1) Suppose condition (1) holds. Let $j_3 \in \{1, 2, 3\} \setminus \{j_1, j_2\}$. From Lemma 4.5, we have $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, and $|\Lambda_{j_3}| \geq 2$. Then we have the following two cases:

Case 1: $|\Lambda_{j_3}| > 2$. Let $\mathcal{I}_0 = \{[R]; R \in \Pi\}$, where $[R] = \{R\}$ for all $R \in \Pi$. Clearly, \mathcal{I}_0 is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2: $|\Lambda_{j_3}| = 2$. Let $\mathcal{I} = \{\Lambda_{j_3}\} \cup \{[R]; R \in \Pi \setminus \Lambda_{j_3}\}$, where $[R] = \{R\}$ for all $R \in \Pi \setminus \Lambda_{j_3}$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

2) Suppose condition (2) holds. Let $A \subseteq \{1, 2, 3\}$ be such that $|\Lambda_j| = 2, \forall j \in A$, and $|\Lambda_j| > 2, \forall j \in \{1, 2, 3\} \setminus A$. Let $\mathcal{I} = \{\Lambda_j; j \in A\} \cup \{[R]; R \in \Pi \setminus (\cup_{j \in A} \Lambda_j)\}$, where $[R] = \{R\}$ for all $R \in \Pi \setminus (\cup_{j \in A} \Lambda_j)$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

3) Suppose condition (3) holds. Let $\mathcal{I} = \{[S_1], [S_2], [S_3], [R]\}$, where $[S_i] = \{S_i\}$ for $i \in \{1, 2, 3\}$ and $[R] = \Pi \setminus \{S_1, S_2, S_3\}$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

4) Without loss of generality, assume $\ell = 1, \ell' = 2$ and $\ell'' = 3$. Let $\mathcal{I} = \{[S_1], [S_2], [S_3]\}$, where $[S_1] = \text{reg}(S_1, S_2) \cup \text{reg}(S_1, S_3) \cup \text{reg}^\circ(S_2, S_3)$, $[S_2] = \{S_2\}$ and $[S_3] = \{S_3\}$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible. ■

Lemma C.3: Suppose $\text{RG}(D^{**})$ has three terminal regions and is terminal separable. If $\text{RG}(D^{**})$ is not feasible, then the condition (C-IR) holds.

Proof: If $\Lambda_{j_1} = \Lambda_{j_2}$ for some $\{j_1, j_2\} \subseteq \{1, 2, 3\}$, then by lemma C.1, we can construct a code \tilde{C}_Π satisfies the conditions of Lemma 4.6. So $\text{RG}(D^{**})$ is feasible. Thus, we assume Λ_1, Λ_2 and Λ_3 are mutually different. Since $\text{RG}(D^{**})$ is not feasible, then by (1), (2) of Lemma C.2, there is a $\{j_1, j_2\} \subseteq \{1, 2, 3\}$ such that

$$|\Lambda_{j_1}| = |\Lambda_{j_2}| = 2 \text{ and } |\Lambda_{j_1} \cap \Lambda_{j_2}| = 1. \quad (10)$$

Let $j_3 \in \{1, 2, 3\} \setminus \{j_1, j_2\}$. By enumerating, we can divide our discussion into the following cases:

Case 1: $\Lambda_{j_1} \cup \Lambda_{j_2} \subseteq \text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_1, S_3) \cup \text{reg}^\circ(S_2, S_3)$. By (10), we can assume

$$\Lambda_{j_1} = \{P_1, P_2\} \text{ and } \Lambda_{j_2} = \{P_0, P_2\}. \quad (11)$$

By Lemma 4.5, $\Lambda_{j_1} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then by proper naming, we can assume

$$P_2 \in \text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \text{ and } P_1 \in \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}). \quad (12)$$

where $\{\ell_1, \ell_2, \ell_3\}$ is a fixed permutation of $\{1, 2, 3\}$. Also, by Lemma 4.5, $\Lambda_{j_2} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then for P_0 , we have the following subcases:

Case 1.1: $P_0 \in \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})$. We can further divide this case into the following two subcases:

Case 1.1.1: $\Lambda_{j_3} \cap (\text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})) \neq \emptyset$. By (4) of Lemma C.2, $\text{RG}(D^{**})$ is feasible.

Case 1.1.2: $\Lambda_{j_3} \cap (\text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})) = \emptyset$. Then $\Lambda_{j_3} \subseteq \text{reg}(S_{\ell_1}, S_{\ell_3}) \cup \{S_{\ell_2}\}$. Moreover, since by Lemma 4.5, $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then either $\Lambda_{j_3} = \{S_1, S_2, S_3\}$ or $\{Q, S_{\ell_2}\} \subseteq \Lambda_{j_3}$ for some $Q \in \text{reg}^\circ(S_{\ell_1}, S_{\ell_3})$.

Let $\mathcal{I} = \{[S_{\ell_1}], [S_{\ell_2}], [S_{\ell_3}], [P_2]\}$, where $[S_{\ell_1}] = \{S_{\ell_1}\}$, $[S_{\ell_2}] = \{S_{\ell_2}\} \cup \text{reg}^\circ(S_{\ell_1}, S_{\ell_3})$, $[S_{\ell_3}] = \{S_{\ell_3}\}$ and $[P_2] = \text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 1.2: $P_0 \in \text{reg}^\circ(S_{\ell_1}, S_{\ell_3})$. This case can be further divided into the following subcases:

Case 1.2.1: $|\Lambda_{j_3}| = 3$ or $\Lambda_{j_3} \subseteq \{P_0, P_1, P_2\}$. Let $\mathcal{I} = \{[P_2]\} \cup \{[R]; R \in \Pi \setminus [P_2]\}$, where $[P_2] = \{P_0, P_1, P_2\}$ and $[R] = \{R\}, \forall R \in \Pi \setminus [P_2]$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 1.2.2: $|\Lambda_{j_3}| = 2$ and $\Lambda_{j_3} \cap \{P_0, P_1, P_2\} = \emptyset$. Assume $\Lambda_{j_3} = \{P_3, P_4\}$. Let $\mathcal{I} = \{[P_2], [P_3]\} \cup \{[R]; R \in \Pi \setminus ([P_2] \cup [P_3])\}$, where $[P_2] = \{P_0, P_1, P_2\}$, $[P_3] = \{P_3, P_4\}$ and $[R] = \{R\}, \forall R \in \Pi \setminus ([P_2] \cup [P_3])$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 1.2.3: $|\Lambda_{j_3}| = 2$ and $\Lambda_{j_3} \cap \{P_0, P_1, P_2\} = \{P_2\}$. By (4) of Lemma C.2, $\text{RG}(D^{**})$ is feasible.

Case 1.2.4: $|\Lambda_{j_3}| = 2$ and $\Lambda_{j_3} \cap \{P_0, P_1, P_2\} = \{P_1\}$ (or $\{P_0\}$). By proper naming, we can assume $\Lambda_{j_3} = \{P_1, P_3\}$, where $P_3 \notin \{P_0, P_1, P_2\}$. If $P_3 \neq S_\ell, \forall \ell \in \{1, 2, 3\}$, then by (3) of Lemma C.2, $\text{RG}(D^{**})$ is feasible. So we assume $P_3 = S_\ell$ for some $\ell \in \{1, 2, 3\}$. Since $P_1 \in \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})$ and, by Lemma 4.5, $\Lambda_{j_3} = \{P_1, P_3\} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then $P_3 = S_{\ell_1}$. Let $j_3 = 1, j_1 = 2, j_2 = 3$ and $\ell_i = i$ ($i = 1, 2, 3$). Then the condition (C-IR) holds.

Case 2: There is an $\ell_1 \in \{1, 2, 3\}$ such that $S_{\ell_1} \in \Lambda_{j_1} \cup \Lambda_{j_2}$. Let $\{\ell_2, \ell_3\} = \{1, 2, 3\} \setminus \{\ell_1\}$. We can further divide this case into the following subcases:

Case 2.1: $S_{\ell_1} \in \Lambda_{j_1} \cap \Lambda_{j_2}$. By proper naming, we assume

$$\Lambda_{j_1} = \{S_{\ell_1}, P_1\} \text{ and } \Lambda_{j_2} = \{S_{\ell_1}, P_2\}. \quad (13)$$

Since, by Lemma 4.5, $\Lambda_{j_1}, \Lambda_{j_2} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then we have

$$P_1, P_2 \in \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}). \quad (14)$$

If $\Lambda_{j_3} \cap \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}) \neq \emptyset$, then by (4) of Lemma C.2, $\text{RG}(D^{**})$ is feasible. So we assume $\Lambda_{j_3} \cap \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}) = \emptyset$. Then

$$\Lambda_{j_3} \subseteq \text{reg}(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}(S_{\ell_1}, S_{\ell_3}). \quad (15)$$

We have the following two subcases:

Case 2.1.1: $\Lambda_{j_3} \cap (\text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}^\circ(S_{\ell_1}, S_{\ell_3})) \neq \emptyset$. Without loss of generality, assume $Q_1 \in \Lambda_{j_3} \cap \text{reg}^\circ(S_{\ell_1}, S_{\ell_2})$. Since, by Lemma 4.5, $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2}), \forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then there is a $Q_2 \in \text{reg}(S_{\ell_1}, S_{\ell_3}) \setminus \{S_{\ell_1}\}$ such that $Q_2 \in \Lambda_{j_3}$. Let $\mathcal{I} = \{[S_{\ell_1}], [S_{\ell_3}]\} \cup \{[R]; R \in \Pi \setminus ([S_{\ell_1}] \cup [S_{\ell_3}])\}$, where $[S_{\ell_1}] = \{S_{\ell_1}\} \cup \text{reg}(P_1, P_2)$, $[S_{\ell_3}] = \{Q_1\} \cup \text{reg}(S_{\ell_1}, S_{\ell_3}) \setminus \{S_{\ell_1}\}$ and $[R] = \{R\}, \forall R \in \Pi \setminus ([S_{\ell_1}] \cup [S_{\ell_3}])$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.1.2: $\Lambda_{j_3} \cap (\text{reg}^\circ(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}^\circ(S_{\ell_1}, S_{\ell_3})) = \emptyset$. By (15), $\Lambda_{j_3} \subseteq \{S_1, S_2, S_3\}$. Since, by Lemma 4.5, $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$, $\forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then $\Lambda_{j_3} = \{S_1, S_2, S_3\}$. Let $\mathcal{I} = \{[S_{\ell_1}] \cup [R]; R \in \Pi \setminus [S_{\ell_1}]\}$, where $[S_{\ell_1}] = \{S_{\ell_1}\} \cup \text{reg}(P_1, P_2)$ and $[R] = \{R\}$, $\forall R \in \Pi \setminus [S_{\ell_1}]$. Clearly, \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.2: $S_{\ell_1} \notin \Lambda_{j_1} \cap \Lambda_{j_2}$. Since $S_{\ell_1} \in \Lambda_{j_1} \cup \Lambda_{j_2}$, then by (10) and proper naming, we can assume $\Lambda_{j_1} = \{S_{\ell_1}, P_1\}$, $\Lambda_{j_2} = \{P_1, P_2\}$. Since, by Lemma 4.5, $\Lambda_{j_1}, \Lambda_{j_2} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$, $\forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$, then $P_1 \in \text{reg}^\circ(S_{\ell_2}, S_{\ell_3})$ and, by proper naming, we can assume $P_2 \in \text{reg}^\circ(S_{\ell_1}, S_{\ell_2})$, where $\{\ell_2, \ell_3\} = \{1, 2, 3\} \setminus \{\ell_1\}$. Let $j_3 \in \{1, 2, 3\} \setminus \{j_1, j_2\}$. If $\Lambda_{j_3} \cap \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}) \neq \emptyset$, then by (4) of Lemma C.2, $\text{RG}(D^{**})$ is feasible. So we assume $\Lambda_{j_3} \cap \text{reg}^\circ(S_{\ell_2}, S_{\ell_3}) = \emptyset$. Then

$$\Lambda_{j_3} \subseteq \text{reg}(S_{\ell_1}, S_{\ell_2}) \cup \text{reg}(S_{\ell_1}, S_{\ell_3}). \quad (16)$$

Now, suppose

$$\Lambda_{j_3} \not\subseteq \text{reg}(S_{\ell_1}, P_2) \cup \text{reg}(S_{\ell_1}, S_{\ell_3}). \quad (17)$$

We shall prove $\text{RG}(D^{**})$ is feasible. We have the following three subcases:

Case 2.2.1: $\Lambda_{j_3} \cap \text{reg}(S_{\ell_1}, P_2) \neq \emptyset$. Since, by Lemma 4.5, $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$ for all $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, then by (16), $\Lambda_{j_3} \cap (\text{reg}(S_{\ell_1}, S_{\ell_3}) \setminus \{S_{\ell_1}\}) \neq \emptyset$. Moreover, by (17), $\Lambda_{j_3} \cap (\text{reg}(S_{\ell_1}, S_{\ell_2}) \setminus \text{reg}(S_{\ell_1}, P_2)) \neq \emptyset$. Let $\mathcal{I} = \{[S_{\ell_1}]\} \cup \{[R]; R \in \Pi \setminus [S_{\ell_1}]\}$, where $[S_{\ell_1}] = \text{reg}(S_{\ell_1}, P_2) \cup \{P_1\}$ and $[R] = \{R\}$, $\forall R \in \Pi \setminus [S_{\ell_1}]$. Then \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

Case 2.2.2: $\Lambda_{j_3} \cap \text{reg}(S_{\ell_1}, P_2) = \emptyset$ and $|\Lambda_{j_3}| \geq 3$. As in Case 2.2.1, we can prove \mathcal{S} is regular.

Case 2.2.3: $\Lambda_{j_3} \cap \text{reg}(S_{\ell_1}, P_2) = \emptyset$ and $|\Lambda_{j_3}| = 2$. Since, by Lemma 4.5, $\Lambda_{j_3} \not\subseteq \text{reg}(S_{i_1}, S_{i_2})$, $\forall \{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then by Equations (16), (17) and proper naming, we can assume $\Lambda_{j_3} = \{P_3, P_4\}$, where $P_3 \in \text{reg}(S_{\ell_1}, S_{\ell_2}) \setminus (\text{reg}(S_{\ell_1}, P_2) \cup \{S_{\ell_2}\})$ and $P_4 \in \text{reg}(S_{\ell_1}, S_{\ell_3}) \setminus \{S_{\ell_1}\}$. Let $\mathcal{I} = \{[S_{\ell_1}], [P_3]\} \cup \{[R]; R \in \Pi \setminus [S_{\ell_1}] \cup [P_3]\}$, where $[S_{\ell_1}] = \text{reg}(S_{\ell_1}, P_2) \cup \{P_1\}$, $[P_3] = \{P_3, P_4\}$ and $[R] = \{R\}$, $\forall R \in \Pi \setminus [S_{\ell_1}] \cup [P_3]$. Then \mathcal{I} is a partition of Π and is compatible. By Lemma 4.13, $\text{RG}(D^{**})$ is feasible.

So for case 2.2, if $\text{RG}(D^{**})$ is not feasible, then $\Lambda_{j_3} \subseteq \text{reg}(S_{\ell_1}, P_2) \cup \text{reg}(S_{\ell_1}, S_{\ell_3})$. Let $\ell_i = j_i = i$, $i = 1, 2, 3$. Then the condition (C-IR) holds.

Combining the discussions for all cases above, we can conclude that if $\text{RG}(D^{**})$ is not feasible, then the condition (C-IR) holds. ■

Lemma C.4: Suppose $\text{RG}(D^{**})$ has three terminal regions and is terminal separable. If the condition (C-IR) holds, then $\text{RG}(D^{**})$ is not feasible.

Proof: We prove this lemma by contradiction. For this purpose, we suppose $\text{RG}(D^{**})$ is feasible and the condition (C-IR) holds. Then there is a code $\tilde{C}_\Pi = \{d_R; R \in \Pi\} \subseteq \mathbb{F}^3$ satisfying conditions of Lemma 4.6. Since $P_1 \in \text{reg}^\circ(S_2, S_3)$, then by Definition 3.7 and condition (2) of Lemma 4.6, we

have

$$d_{P_1} \in \langle \alpha_2, \alpha_3 \rangle.$$

Moreover, since $\Lambda_1 = \{S_1, P_1\}$, then by conditions (1), (3) of Lemma 4.6, we have $\bar{\alpha} \in \langle \alpha_1, d_{P_1} \rangle$. So

$$d_{P_1} \in \langle \alpha_1, \bar{\alpha} \rangle \cap \langle \alpha_2, \alpha_3 \rangle = \langle \alpha_2 + \alpha_3 \rangle.$$

Similarly, since $\Lambda_2 = \{P_1, P_2\}$ and $P_2 \in \text{reg}^\circ(S_1, S_2)$, then

$$d_{P_2} \in \langle d_{P_1}, \bar{\alpha} \rangle \cap \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_1 \rangle.$$

By Definition 3.7 and condition (2) of Lemma 4.6, $d_R \in \langle \alpha_1 \rangle$ for all $R \in \text{reg}(S_1, P_2)$ and $d_R \in \langle \alpha_1, \alpha_3 \rangle$ for all $R \in \text{reg}(S_1, S_3)$. Since $\Lambda_3 \subseteq \text{reg}(S_1, P_2) \cup \text{reg}(S_1, S_3)$, then

$$\langle d_R; R \in \Lambda_3 \rangle \subseteq \langle \alpha_1, \alpha_3 \rangle.$$

By condition (3) of Lemma 4.6, we have $\bar{\alpha} \in \langle d_R; R \in \Lambda_3 \rangle \subseteq \langle \alpha_1, \alpha_3 \rangle$, a contradiction. Thus, we can conclude that if the condition (C-IR) holds, then $\text{RG}(D^{**})$ is not feasible. ■

Now, we can prove Theorem 4.16.

Proof of Theorem 4.16: By enumerating, one of the following three cases hold:

Case 1: $\Omega_{1,2,3} \neq \emptyset$. Then there is a $Q \in D^{**} \setminus \Pi$ such that $Q \rightarrow T_i$, $i = 1, 2, 3$. Similar to what we did in Remark 4.4, we can first construct a code on the set $\{R \in D^{**}; R \rightarrow Q\}$ such that $d_Q = \bar{\alpha}$. Then for all R such that $Q \rightarrow R \rightarrow T_i$ for some $i \in \{1, 2, 3\}$, let $d_R = \bar{\alpha}$. By this construction, we obtain a solution of $\text{RG}(D^{**})$. So $\text{RG}(D^{**})$ is feasible.

Case 2: $\Omega_{1,2,3} = \emptyset$ and $\Omega_{i_1, i_2} \neq \emptyset$ for some $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. Then there is a $Q \in D^{**} \setminus \Pi$ such that $Q \rightarrow T_{i_1}$ and $Q \rightarrow T_{i_2}$. Let $i_3 \in \{1, 2, 3\} \setminus \{i_1, i_2\}$. We can view T_{i_3} and Q as two terminal regions and, by Lemma C.1, we can construct a code on the set $\{R \in D^{**}; R \rightarrow Q \text{ or } R \rightarrow T_{i_3}\}$ such that $d_Q = d_{T_{i_3}} = \bar{\alpha}$. Moreover, for all R such that $Q \rightarrow R \rightarrow T_{i_1}$ or $Q \rightarrow R \rightarrow T_{i_2}$, let $d_R = \bar{\alpha}$. Then we obtain a solution of $\text{RG}(D^{**})$. So $\text{RG}(D^{**})$ is feasible.

Case 3: $\text{RG}(D^{**})$ is terminal separable. By Lemma C.3 and C.4, $\text{RG}(D^{**})$ is not feasible if and only if, by proper naming, the condition (C-IR) holds.

By the above discussion, we proved Theorem 4.16. ■